

Nested Topological Order

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We introduce the concept of nested topological order in a class of exact quantum lattice Hamiltonian models with non-abelian discrete gauge symmetry. The topological order present in the models can be partially destroyed by introducing a gauge symmetry reduction mechanism. When symmetry is reduced in several islands only, this imposes boundary conditions to the rest of the system giving rise to topological ground state degeneracy. This degeneracy is related to the existence of topological fluxes in between islands or, alternatively, hidden charges at islands. Additionally, island deformations give rise to an extension of topological quantum computation beyond quasiparticles.

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The concept of topological orders [1] offers the possibility of finding new states of matter with a common picture of string-net condensation [2] and other variants thereof [3]. They correspond to examples of long range entanglement in quantum many-body systems where those correlations emerge in quantum states that are encoded in non-local degrees of freedom of topologically ordered systems. Their global properties are the source for yet another application as the suitable systems to implement topological quantum computation [4, 5, 6, 7], a form of fault-tolerant quantum computation intrinsically resistant to the debilitating effects of local noise. Quantum field theories with an spontaneous symmetry breaking mechanism of a continuous gauge group down to a discrete group have been proposed as a scenario for realizing their physics [8, 9, 10, 11, 12, 13, 14].

In this paper we introduce the concept of nested topological order in a class of quantum lattice Hamiltonians. Our starting point are the family of Kitaev's models [4], which are labeled by a discrete gauge group. Such models can be modified [15] introducing an explicit symmetry breaking mechanism. Our aim is to study the effect of 'nesting' subsystems with a reduced symmetry inside systems with the complete gauge symmetry. We will consider a topologically ordered system divided in two regions, say A and C , and show that it is possible to partially destroy the topological order in region C in such a way that this imposes boundary conditions to the subsystem A . The system C can take the form of several islands, which is why we talk about 'nested' topological order. The boundary conditions induce a topological ground state degeneracy which is due to the possible values of certain fluxes in between islands. As we will see, the values of these fluxes correspond to the types of domain walls that exist in C . If we allow the region C to be deformed, then islands can be initialized, braided and fused, giving an interesting extension of the ideas of topological quantum computation beyond quasiparticles.

The models that we consider are string-net condensates in a 2D lattice [1], [2]. The configurations of the lattice are regarded as string-net states: a collection of labeled strings meeting at branching points. A string-net is closed if certain conditions hold at branching points

and there are no loose ends. The ground state is a superposition of all possible deformations of such closed string-nets, and excited states correspond to configurations with loose ends: quasiparticle excitations appear at the ends of strings. Now, to such system Hamiltonians we can add string tension terms, which penalize with a higher energy those configurations with longer strings. As such terms get more important with respect to the original ones, longer strings become less relevant in the ground state and finally the topological order is destroyed as excitations get confined. Alternatively, we can add suitable terms so that only part of the topological order is destroyed. This is in fact the case for the Hamiltonians $H_G^{N,M}$ that we consider (1), which are labeled with a discrete group G and two subgroups $N \subset M \subset G$, with N abelian and normal in G . If $N = 1$ and $M = G$, we have the original topologically ordered models with gauge group G considered by Kitaev [4]. Otherwise, the gauge symmetry is reduced the quotient group $G' = M/N$. In particular, if $N = M$ the topological order is completely destroyed.

Topological phases. The systems of interest are constructed from a two-dimensional orientable lattice, of arbitrary shape. At every edge of the lattice we place a qudit, a $|G|$ -dimensional quantum system with Hilbert space \mathcal{H}_G and a basis $|g\rangle$ labeled with the elements of G . The Hamiltonians read as follows[15]

$$H_G^{N,M} := - \sum_{v \in V} A_v^M - \sum_{f \in F} B_f^N - \sum_{e \in E} (T_e^M + L_e^N), \quad (1)$$

where the sums run over the set of vertices V , faces F and edges E . Explicit expressions for the terms in (1) will be given below, but before that, we will discuss their physical content. First, all the terms are projectors and commute with each other, so that the ground state is described by conditions of the form $P|\text{GS}\rangle = |\text{GS}\rangle$ with P either a vertex, face or edge operator. Excitations are gapped and localized; they correspond to violations of the previous conditions and so can be related to vertices, faces and edges; they are regarded respectively as electric, magnetic and domain wall excitations.

We first recall the case $H_G := H_G^{1,G}$ [4]. For non-Abelian groups G , vertex and face excitations are inter-

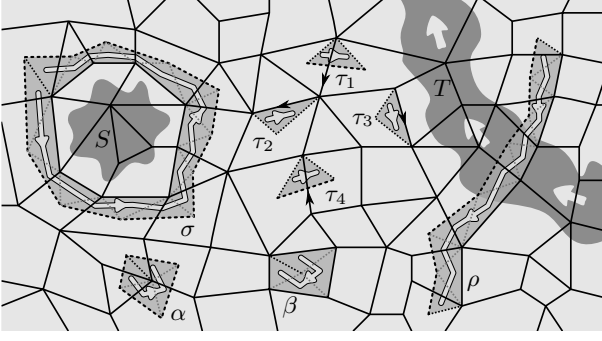


FIG. 1: Examples of lattice constructions. Although all the edges must be oriented, only the orientation of some of them is shown. The $\tau_i, i = 1, 2, 3, 4$ are triangles; the light thick arrow shows their orientation. τ_1 and τ_4 are dual, the others are direct. σ is a closed ribbon; the projectors $K_\sigma^{R,C}$ give the charge in the region S that σ encloses. ρ is an open ribbon; the projectors $J_\sigma^{R,T}$ give the domain wall flux in the region T in the direction of the arrows. α and β are minimal closed ribbons, enclosing respectively a single vertex and face.

related and the excitation types, labeled as (R, C) , are dyons: C , the magnetic part, is a conjugacy class of G and R , the electric part, is an irrep of \mathbf{N}_C , the group $\mathbf{N}_C := \{g \in G | gr_C = r_C g\}$, where r_C is some chosen element of C . These charges have a topological nature: if there are several excited spots in the system, far apart from each other, there exist certain global degrees of freedom which cannot be accessed through local operators.

In the general case $H_G^{N,M}$ there are two new phenomena, quasiparticle condensation and the appearance of domain wall excitations. The latter have an energy proportional to their length and can be labeled by pairs (R, T) , with $T \in M \backslash G/M$ and R an induced representation in M of an irrep of the group $\mathbf{N}_T := \{m \in M | mr_T M = r_T M\}$, where r_T is some chosen element of T . Thus there exists a flux related to domain walls, with values (R, T) ; it is conserved in the absence of quasiparticle excitation, so that domain walls only can end at them. As for condensation, we will comment upon it below.

Ribbon operators. In order to motivate the introduction of ribbon operators, we first note that dyons, the excitations of our system, are located at vertex-face pairs, which are called sites. In Fig. 1 sites are represented as dotted lines connecting the vertex to the center of the face. The basic connectors between sites are triangles: just as an edge connects two vertices, triangles connects two sites. A direct (dual) triangle τ is composed by two sites and a direct (dual) edge e_τ , see Fig. 1. Triangles can be concatenated to form ribbons connecting distant sites. Ribbons are open if they connect disjoint sites and closed if their ends coincide. The point is that it is possible to attach to each ribbon ρ certain operators $F_\rho^{h,g}$, $h, g \in G$, which are very well suited to represent excited states. For example, any state with only two dyons is a

linear combination of the states $F_\rho^{h,g}|\text{GS}\rangle$, with ρ any ribbon connecting the sites where the dyons are located[4]. In fact, one can consider that ribbon operators represent a process in which a particle-antiparticle is created in one end of the ribbon and one of them is moved to the other end.

In order to describe ribbon operators, we start with triangles, which are the smallest ribbons. Recall that a triangle is formed by two sites and one edge, direct or dual. Triangle operators act on the qudit attached to that edge, and the action depends on the orientation of the edge and the type of the triangle. The four possible cases are illustrated in Fig. 1. With the notation of that figure, we have $F_{\tau_1}^{h,g} = \delta_{g,1} \sum_k |hk\rangle\langle k|$, $F_{\tau_2}^{h,g} = |g^{-1}\rangle\langle g^{-1}|$, $F_{\tau_3}^{h,g} = |g\rangle\langle g|$ and $F_{\tau_4}^{h,g} = \delta_{g,1} \sum_k |kh^{-1}\rangle\langle k|$, where the sums run over G . Then if ρ is a ribbon formed by the concatenation of the ribbons ρ_1 and ρ_2 , we set $F_\rho^{h,g} = \sum_k F_{\rho_1}^{h,k} F_{\rho_2}^{k^{-1}hk, k^{-1}g}$. The terms in the Hamiltonians (1) are built from ribbon operators. Let $F_\rho^{UV} := |U|^{-1} \sum_{u \in U} \sum_{v \in V} F_\rho^{u,v}$ for any subgroups $U, V \subset G$. Then $A_v^M := F_\alpha^{NG}$, $B_f^N := F_\beta^{1N}$, $T_e^M := F_\tau^{1M}$ and $L_e^N := F_{\tau'}^{NG}$, with α and β suitable minimal closed ribbons as in Fig. 1 and τ (τ') a direct (dual) triangle with $e = e_\tau$.

Ribbon operators commute with all the vertex operators A_v^G and face operators B_f^1 , except with those at their ends. Moreover, they can be characterized by this property[15]. This suggests considering, for closed ribbons σ , those ribbon operators which commute with all vertex and face operators, so that they ‘forget’ the single end of σ . It turns out that a linear basis for such operators is given by a family of projectors $K_\sigma^{R,C}$, labeled with the charge types (R, C) of the system H_G . In fact, if σ is a boundary ribbon, that is, a closed ribbon enclosing certain region S as in Fig. 1, then $K_\sigma^{R,C}$ projects out those states with total topological charge (R, C) in S . As a result, the ground state of H_G can be described by the conditions

$$F_\sigma^{G1}|\psi\rangle = |\psi\rangle, \quad (2)$$

which must hold for all boundary ribbon σ . This amounts to impose that all disc shaped regions must have trivial charge because $K_\sigma^{e1} = F_\sigma^{G,1}$, where e is the identity representation. In systems with Hamiltonian $H_G^{N,M}$ we can use the projectors K_σ^{RC} to describe condensation. Namely, for some charges [15] we have a ground state expectation value $\langle K_\sigma^{RC} \rangle > 0$ for any boundary ribbon σ , showing that there exist a non-zero probability of finding such charges in a given region.

Domain wall types can be obtained in a similar fashion in systems with Hamiltonian $H_G^{N,M}$. For any open ribbons ρ , those ribbon operators that commute with all vertex operators A_v^M and face operators B_f^N are linear combinations of certain projectors $J_\rho^{R,T}$, with (R, T) a domain wall type. If ρ crosses an area with domain wall excitations then $J_\rho^{R,T}$ projects out those states with total domain wall flux (R, T) across ρ . For example, in Fig. 1

ρ will measure the flux of the excited region T in the direction of the white arrows.

The ground states of (1) can also be described in terms of conditions for ribbon operators, in particular by

$$F_{\sigma}^{MN}|\text{GS}\rangle = |\text{GS}\rangle, \quad F_{\rho}^{NM}|\text{GS}\rangle = |\text{GS}\rangle, \quad (3)$$

where σ and ρ are arbitrary boundary and open ribbons, respectively. The first condition is related to vertex and face excitations, and the second to edge excitations.

Nested phases. We are now in position to discuss a more complicated system. In particular, we want to consider a surface divided in two regions of arbitrary shape, A and C , plus a third region B which is just a thick boundary separating them, included so that the Hamiltonian does not have to change abruptly from A to C . The idea is to have a local Hamiltonian such that conditions (2) are satisfied in A , conditions (3) in C and the conditions

$$F_{\sigma}^{NN}|\text{GS}\rangle = |\text{GS}\rangle, \quad (4)$$

with σ an arbitrary boundary ribbon, in the whole system. The last condition is needed to ensure that domain wall flux is preserved through region B , a key ingredient of our construction as we will see. The ground state of the Hamiltonian $H_0 := -\sum_v A_v^N - \sum_f B_f^N$ is described precisely by (4). In addition, H_0 commutes with H_G , H_G^{NM} . Indeed, a Hamiltonian of the form $H' = H_G + \lambda H_0$, $\lambda \geq 0$, only differs from H_G in the gap for some excitations, and the same is true for H_G^{NM} . The Hamiltonian that we want to consider takes the form $H = H_0 + \lambda H_G + \mu H_G^{NM}$, where $\lambda, \mu \geq 0$ vary spatially so that $\lambda = 1$ and $\mu = 0$ in A and $\lambda = 0$ and $\mu = 1$ in C . If we take $\lambda\mu = 0$, the ground state has the desired properties but there exists some local degeneracy at B . This local degeneracy can be lifted if λ and μ are allowed to overlap, but on the other hand if the overlap is too big, it could produce a level crossing taking the ground state of H out of that of H_0 , which spoils conditions (4).

Quasiparticle dilution. Our aim is to understand the effects of the nested region C on the topologically ordered region A . A first effect is the possibility to locally create or destroy single quasiparticle excitations in the vicinity of the A - C border, something prohibited in systems with Hamiltonian H_G due to charge conservation. In terms of ribbon operators, this is reflected in the fact that for any ρ_1 connecting C to A , as the one in Fig. 2(a), a state of the form $\sum_{m \in M} F_{\rho_1}^{mn, mg} |\text{GS}\rangle$, $n \in N$, contains no excitation at C . In terms of quasiparticle processes, this corresponds to create a particle-antiparticle pair in A and then move one of them into C , where it disappears because it is condensed.

Domain wall dilution. A second effect is related to the existence of domain walls in region C . Consider again a ribbon ρ_2 connecting C to A , see Fig. 2(a). Some of the states of the form $|\psi\rangle = \sum_{h,g} c_{h,g} F^{h,g} |\text{GS}\rangle$, $c_{h,g} \in \mathbb{C}$, will contain edge excitations all along the portion of ρ_2 contained in C , for example those with $c_{h,g} \neq 0$ for some

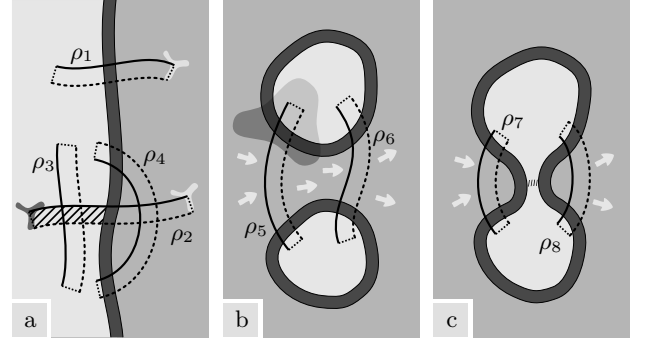


FIG. 2: In this figure regions A , B and C are shaded respectively with medium, dark and light gray. Ribbons $\rho_i, i = 1, \dots, 8$ are displayed as pairs of solid and dashed parallel lines which correspond respectively to their direct and dual edges. Light spots at the end of ribbons represent excitations in A and the dark one an excitation in C . The striped areas are domain wall excitations. (a) Due to condensation, suitable ribbon operators attached to ρ_1 will create an excitation in A but no excitation in C . Ribbon operators attached to ρ_2 can create a domain wall excitation in C . The resulting state ψ is such that $J_{\rho_3}^{R,T}|\psi\rangle = J_{\rho_4}^{R,T}|\psi\rangle$. (b) Both ρ_5 and ρ_6 measure the flux in between the islands. If O is an operator with support in the shaded area it cannot change the flux. (c) If the previous islands are deformed till they fuse, the flux measured by ρ_7 , ρ_8 will remain the same as it was for ρ_5 , ρ_6 . If it is nontrivial, opposite border charges are present at the sides of the meeting point.

$g \in G, h \notin M$. These excitations form a domain wall, to which we can relate a type or flux given by the projector $J_{\rho_3}^{RC}$, where ρ_3 is a ribbon that lies in C and crosses the domain wall, see Fig. 2(a). Such a ribbon can be deformed without crossing any quasiparticle excitation onto another ribbon ρ_4 that only has its endpoints in C and thus avoids the domain wall, so that $J_{\rho_3}^{R,T}|\psi\rangle = J_{\rho_4}^{R,T}|\psi\rangle$ due to (4). Both ribbon operators are measuring the same domain wall flux. However, in the case of ρ_4 the flux is being measured in A , where the domain wall gets diluted as it turns into a condensed string. Note that $J_{RC}^{R,T}$ cannot detect changes in the interior of C . In this regard, if we restrict our attention to region A , domain wall flux projectors from ribbons like ρ_4 , that is, which enclose a portion of the A - C border, can be related to charges (R, T) that lie in that piece of the A - C border.

Induced topological fluxes. Things get even more interesting if we consider that C consists of several disjoint parts. For example, consider a plane and choose as the region C two islands C_1 and C_2 , see Fig. 2(b). Now instead of considering a domain wall flux coming out from a region of B (such as the one measured by ρ_4 in Fig. 2(a)), we consider the flux in between the two islands (as indicated by the arrows in Fig. 2(b)). This is the flux measured by the projectors $J_{\rho_5}^{R,T}$, where ρ_5 is any ribbon that connects the islands, as in Fig. 2(b). The point is that such a flux is a global (topological) prop-

erty as long as the islands are distant. Indeed, measuring the flux requires an operator with a support connecting C_1 and C_2 . And, if an operator changes the flux, its support must loop around C_1 (or C_2). Suppose to the contrary that O is an operator that leaves the ground state invariant and has a support not enclosing C_1 , as the shaded region in Fig. 2(b). Let ρ_6 be another ribbon connecting the islands but lying outside the support of O . Due to (4) we have $J_{\rho_5}^{R,T}|\text{GS}\rangle = J_{\rho_6}^{R,T}|\text{GS}\rangle$, so that $[J_{\rho_5}^{R,T}, O]|\text{GS}\rangle = [J_{\rho_6}^{R,T}, O]|\text{GS}\rangle = 0$ and thus O does not change the flux. Those operators which do change the flux are related to processes in which a particle-antiparticle pair is created, one of them loops around C_1 and they meet again to fuse into a charge that disappears into C_1 .

Topologically protected subsystems. It follows that there exist a topological degeneracy in the ground state, related to the distinct values that the flux in between C_1 and C_2 can take. For example, if $N = M = 1$ the flux can take any value $g \in G$. In general, for a C composed of multiple disconnected regions, the degeneracy of the ground state depends on N , M and the topology of A . Now, it is natural to ask how does this protected space compares with the one due to the existence of several separated excitations in A . In other words, do islands add something new? This can be positively answered through an example: two excitations give no protected subspace [4], but we have just seen the contrary for the case of two islands. Perhaps more dramatically, for abelian groups G the protected subsystem is always trivial whatever the amount of excitations, but this is not the case for islands. Nevertheless, islands can be compared to excitations, in the following sense. An island can hold certain charge values, which can be measured using ribbon operators that enclose the island, as in the case of an excitation. The difference between a charged island and a charged excitation is that the local degrees of freedom of the excitation become global in the case of the island: this is the origin of the additional dimensionality of the protected subsystem.

Braiding. The physics of the system so far has a static nature. If we want to consider the setting as an sce-

nario for quantum computation, then the possibility of dynamically deforming the region C must be included in it. Such deformations need not be strictly adiabatic, but the state should be kept in the subspace defined by conditions (2-4) at all time. We can then braid islands to perform unitary operations, in complete analogy with quasiparticle braiding. It is also natural to enrich the physics by considering islands with different (N, M) labels, increasing the variety of protected subsystems.

Fusion. We must consider also the analogue of the quasiparticle fusion processes, which is the way in which measurements are carried out in topological quantum computation. There are two natural ways in which global degrees of freedom can be made local. The first is to decrease the size of an island till it disappears leaving a small charged region. The outcome of such a process is the charge, which can be measured but not changed locally. The second way is closer to the idea of fusion. Indeed, it is also a fusion, but of islands instead of quasiparticles. The idea is depicted in Fig. 2(c). As two islands of the same (N, M) type get closer, some of the ribbon operators connecting them become small and thus the flux between the islands is exposed to local measurements. If we continue the approach till the islands meet, the flux will take the form of a domain wall excitation at the meeting place, as in Fig. 2(c). Due to confinement the domain wall can decay to several smaller walls, but there is something that will not disappear, the two border charges in its ends on B . As explained in the caption of Fig. 2(c), the appearance of this border charges can be seen directly in terms of ribbon operators. Regarding the initialization of the system, reverse processes can be used. That is, if an island is divided in two, the topological flux in between them will be trivial, and if an island is created from the vacuum, it will have trivial charge. In both cases the reason is that topological properties cannot be changed by local processes.

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